BAULKHAM HILLS HIGH SCHOOL

DECEMBER 2013
YEAR 12 TASK 1

Mathematics

General Instructions

- Reading time – 5 minutes
- Working time – 50 minutes
- Write using black or blue pen
- Board-approved calculators may be used
- Show all necessary working in questions 1 to 5
- Marks may be deducted for careless or badly arranged work

Total marks – 50
Exam consists of 4 pages.

This paper consists of ONE section.

Questions 1-5

- Attempt Question 1-5

Topics Tested: Integration and Series with Applications

Question 1 (11 marks) - Start a new page

a) The first term of an arithmetic progression is 3 and the second term is 9.

Find a) the 10^{th} term **2**

b) the sum of 10 terms. **2**

b) A geometric progression has an nth term of $T_n = 4 \times 3^{-n}$. **1**

Find the ratio $\frac{T_{100}}{T_{98}}$

c) Find **2**

(i) $\int \frac{x^3+2}{x^2}\,dx$

(ii) $\int (4-7x)^5 dx$ **2**

d) Evaluate **2**

$$\int_1^9 6 - \frac{8}{\sqrt{x}}\,dx$$

Question 2 (10 marks) - Start a new page

a) (i) Draw a neat sketch of $y = x^3$ **1**

(ii) Find the area bounded by the curve and the x axis between $x = -1$ and $x = 2$ **3**

b) Given that $\frac{dy}{dx} = ax - 4$ find the equation of the curve, when $x = 1, \frac{dy}{dx} = 5$ and $y = -2$. **3**

c) The sum of the first five terms of an arithmetic sequence is 10 and the seventh term is three times the fourth term. **3**

Find the first three terms of the sequence.

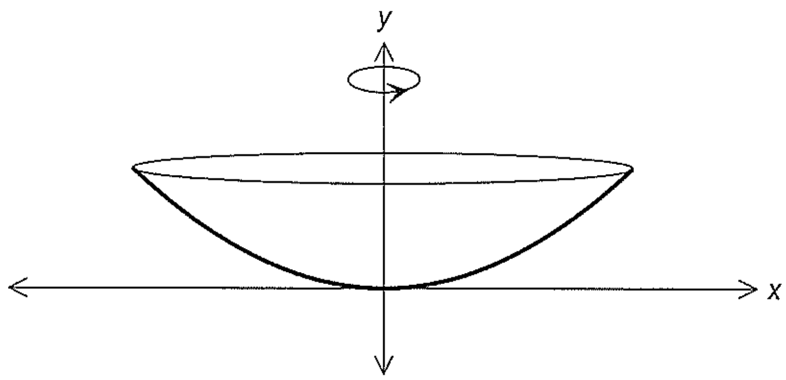
Question 3 (9 marks) - **Start a new page**

a) Evaluate 2

$$\sum_{n=2}^{5}(2n^2+n)$$

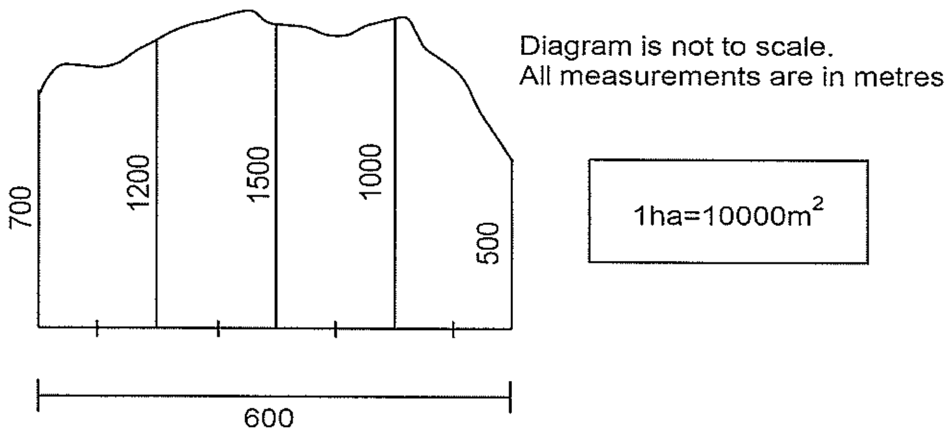
b) A fruit bowl has a shape obtained by rotation part of the parabola $y=\frac{x^2}{2}$ around the y axis. 3

The bowl is $10cm$ deep.



Find the volume of the fruit bowl.

c) A farm paddock is to be sprayed with fertilizer and an approximation of the area is needed in hectares. By using Simpson's Rule, with five function values, find an approximation of the area represented in the diagram. 2



d) If $\int_1^2 h(x)\,dx=6$ find k such that $\int_1^2 [h(x)+kx^2]\,dx=27$ 2

Question 4 **(10 marks) - Start a new page**

a) Given the infinite series

$$\frac{x}{3}+\frac{2x^2}{9}+\frac{4x^3}{27}+\cdots$$

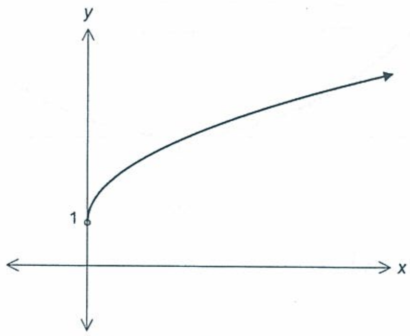
(i) Show that it is a geometric series **1**

(ii) Find the value(s) of x such that the series has a limiting sum. **2**

(iii) Find the limiting sum in terms of x, in simplest form. **1**

b) (i) Show that the curve $y = 1 + \sqrt{x}$ and the lines $y = 7 - x$ meet at the point (4,3) **1**

(ii) Copy the diagram given for $y = 1 + \sqrt{x}$ and sketch $y = 7 - x$ onto your diagram. **1**



(iii) Shade the area between the y axis, $y = 7 - x$ and $y = 1 + \sqrt{x}$ **1**

(iv) Find the value of this area. **3**

Question 5 **(10 marks) - Start a new page**

a) A woman borrows \$48000 from Joe, the money lender, to buy a new car. She plans to pay it off with equal monthly instalments over 5 years at 15%pa, Let A_n be the amount owing after n months and M the amount of each repayment. compounding monthly.

(i) Show that $A_3 = 48000(1.0125)^3 - M(1 + 1.0125 + 1.0125^2)$ **2**

(ii) Show that the expression to evaluate M can be given as **2**

$$M = 48000(1.0125)^{60} \times \frac{0.0125}{(1.0125^{60} - 1)}$$

(iii) Hence evaluate the monthly instalment, M. **1**

(iv) How much interest did she pay? **2**

b) For a given arithmetic sequence, the ratio of the sum of the first m terms to the sum of the first n terms is $m^2 : n^2$. **3**

Prove that the ratio of the m^{th} term to the n^{th} term is $(2m - 1) : (2n - 1)$ where $m \neq n$

- END OF PAPER -

Yr11 2 unit Solutions Dec 2013

Question 1.

a) $a = 3 \quad d = 6$

i) $T_n = a + (n-1)d$

$T_{10} = 3 + 9 \times 6$

$= \underline{57}$

ii) $S_n = \frac{n}{2}(a + l)$

$S_{10} = \frac{10}{2}(3 + 57)$

$= 5 \times 60$

$= \underline{300}$

b) $T_n = 4 \times 3^{-n}$

$$\frac{T_{100}}{T_{98}} = \frac{\cancel{4} \times 3^{-100}}{\cancel{4} \times 3^{-98}}$$

$= 3^{-2}$ or $\frac{1}{9}$

c)

i) $\int x + 2x^{-2}\,dx = \frac{x^2}{2} - 2x^{-1} + c$ or $\frac{x^2}{2} - \frac{2}{x} + c$

ii) $\int (4-7x)^5\,dx = -\frac{(4-7x)^6}{42} + c$

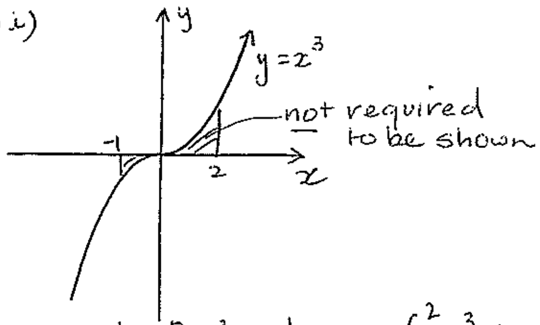
d) $\int_1^9 6 - 8x^{-\frac{1}{2}}\,dx = \left[6x - \frac{8x^{\frac{1}{2}}}{\frac{1}{2}}\right]_1^9$ or $\left[6x - 16x^{\frac{1}{2}}\right]_1^9$

$= (54 - 16 \times 3) - (6 - 16)$

$= \underline{16}$

Question 2.

a) i)



ii) $A = \left| \int_{-1}^{0} x^3\,dx \right| + \int_0^2 x^3\,dx$

$= \left| \left[\frac{x^4}{4}\right]_{-1}^{0} \right| + \left[\frac{x^4}{4}\right]_0^2$

$= \left| 0 - \frac{1}{4} \right| + \left(\frac{16}{4} - 0\right)$

$= 4\frac{1}{4}\,u^2$

b) $\frac{dy}{dx} = ax - 4 \qquad x = 1$

$5 = a - 4$

$a = 9$

$\therefore \frac{dy}{dx} = 9x - 4$

$y = \frac{9x^2}{2} - 4x + c$

$-2 = \frac{9}{2} - 4 + c$

$c = -\frac{5}{2}$

$\therefore y = \frac{9x^2}{2} - 4x - \frac{5}{2}$

c) $S_5 = \frac{5}{2}(2a+4d) = 10$

$10 = \frac{5}{2}(-3d+4d)$

$\frac{20}{5} = d$

$d = 4$

$T_7 = 3T_4$

$a+6d = 3(a+3d)$

$a+6d = 3a+9d$

$-3d = 2a$. (sub into S_5)

$\therefore 2a = -12$

$a = -6$. $\therefore$ the series: $\underline{-6, -2, 2,}$ . .

Question 3.

a) $\sum_{n=2}^{5}(2n^2+n) = 10+21+36+55$

$= \underline{122}$

b) $V = \pi\int x^2 dy$

$= \pi\int_0^{10} 2y\,dy$

$= \pi\left[y^2\right]_0^{10}$

$= 100\pi - 0$

$= 100\pi$ cm

c) $A = \frac{150}{3}\left\{700 + 4\times(1200+1000) + 2\times1500 + 500\right\}$

$= \frac{150}{3} \times 13000$

$= 650\,000$

$= \underline{65}$ hectares.

d) $\int_1 h(x)\,dx = 6$

$\int_1^2 h(x)\,dx + \int_1^2 kx^2\,dx = 27$

$6 + \left[\frac{kx^3}{3}\right]_1^2 = 27$

$\frac{8}{3}k - \frac{k}{3} = 21$

$7k = 63$

$k = 9.$

Question 4.

a) i) $r = \frac{T_2}{T_1} = \frac{T_3}{T_2}$

$\frac{T_2}{T_1} = \frac{2x^2}{9} \div \frac{x}{3}$

$= \frac{2x}{3}$

$\frac{T_3}{T_2} = \frac{4x^3}{27} \times \frac{9}{2x^2}$

$= \frac{2x}{3}$

$\therefore r = \frac{2x}{3}$. as r is consistent the series is a GP.

ii) $|r| < 1$ for a limiting sum

$\left|\frac{2x}{3}\right| < 1$

$\therefore -1 < \frac{2x}{3} < 1$

$-\frac{3}{2} < x < \frac{3}{2}$ or $|x| < \frac{3}{2}$.

iii) $\lim S = \frac{a}{1-r}$

$= \frac{\frac{x}{3}}{1-\frac{2x}{3}}$

$= \frac{\frac{x}{3}}{\frac{3-2x}{3}}$

$= \frac{x}{3-2x}$

Question 5

a) i) $P = \$48000$ or 4.8×10^4 $\quad n = 60 \quad r = \frac{0.15}{12} = 0.0125$

$A_1 = P(1+r) - M$

$= 48000(1.0125) - m$

$A_2 = A_1(1+r) - M$

$= 48000(1.0125)^2 - m(1.0125) - M$

$A_3 = A_2(1+r) - M$

$= 48000(1.0125)^3 - m(1 + 1.0125 + 1.0125^2)$

ii) $A_{60} = 48000(1.0125)^{60} - m(\underbrace{1 + 1.0125 + \cdots 1.0125^{59}}_{S_{60} \text{ of G.P.}})$ $\to a = 1$, $r = 1.0125$

$= 48000(1.0125)^{60} - m\left(\frac{1(1.0125^{60} - 1)}{1.0125 - 1}\right)$

but $A_{60} = 0$

$m\left(\frac{1.0125^{60} - 1}{1.0125 - 1}\right) = 48000(1.0125)^{60}$

$m = 48000(1.0125)^{60} \times \frac{0.0125}{(1.0125^{60} - 1)}$

iii) $M = \underline{\$1141.92}$

iv) $I = 60 \times 1141.92 - 48000$

$= \underline{\$20515.20}$

b) i) $y = 1 + \sqrt{x}$ $\quad y = 7 - x$

sub in $(4,3)$

$y = 1 + \sqrt{4}$ $\quad y = 7 - 4$

$= 3$ ✓ $\quad = 3$ ✓

$\therefore$ curve and line meet at $(4,3)$

$\boxed{\text{or}}$ $\left.\begin{array}{l} y = 1 + \sqrt{x} \\ y = 7 - x \end{array}\right\}$

$1 + \sqrt{x} = 7 - x$

$\sqrt{x} = 6 - x$

$x = 36 - 12x + x^2$

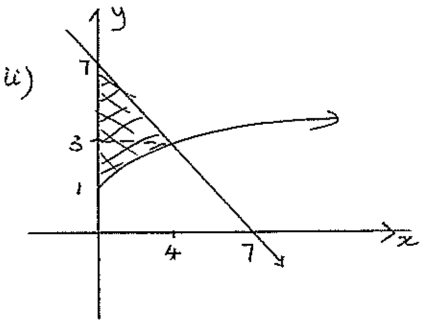
$x^2 - 13x + 36 = 0$

$(x-9)(x-4) = 0$

$x = 4, 9$. $\quad \therefore x = 4 \quad y = 7 - 4 = 3$

$(4,3)$

ii)



(iii) correct shading

iv) $A = \int_3^7 (7-y)\,dy + \int_1^3 (y^2 - 2y + 1)\,dy$

$= \left[7y - \frac{1}{2}y^2\right]_3^7 + \left[\frac{y^3}{3} - y^2 + y\right]_1^3$

$= \left(49 - \frac{49}{2}\right) - \left(21 - \frac{9}{2}\right) + (9 - 9 + 3) - \left(\frac{1}{3} - 1 + 1\right)$

$= \underline{\frac{32}{3}\ u^2 \text{ or } 10\frac{2}{3}\ u^2}$

b) $S_m = \frac{m}{2}(2a + (m-1)d)$ $\qquad S_n = \frac{n}{2}(2a + (n-1)d)$

$m \neq n.$

$S_m : S_m = m^2 : n^2$

ie $\dfrac{\frac{m}{2}(2a + (m-1)d)}{\frac{n}{2}(2a + (n-1)d)} = \dfrac{m^2}{n^2}$

$$\frac{2a + (m-1)d}{2a + (n-1)d} = \frac{m}{n}.$$

$$2an + n(m-1)d = 2am + m(n-1)d.$$

$$2a(n-m) = m(n-1)d - n(m-1)d$$

$$2a(n-m) = \cancel{mn} - md - \cancel{mn} + nd$$

$$2a\cancel{(n-m)} = \cancel{(n-m)}d$$

$$\underline{\underline{2a = d}}.$$

now. $\dfrac{T_m}{T_n} = \dfrac{a + (m-1)d}{a + (n-1)d}$

$$= \frac{a + 2a(m-1)}{a + 2a(n-1)}$$

$$= \frac{a - 2a + 2am}{a - 2a + 2an}$$

$$= \frac{2am - a}{2an - a}$$

$$= \frac{2m-1}{2n-1}.$$